Real-time Software for Control

H-infinity Control

Robustness

- Uncertainty modeling
- Additive uncertainty

$$\tilde{G}(s) = G(s) + \Delta_a(s) \qquad \Delta_a(s) = \tilde{G}(s) - G(s)$$

- Multiplicative uncertainty

$$\tilde{G}(s) = [1 + \Delta_m(s)]G(s)$$

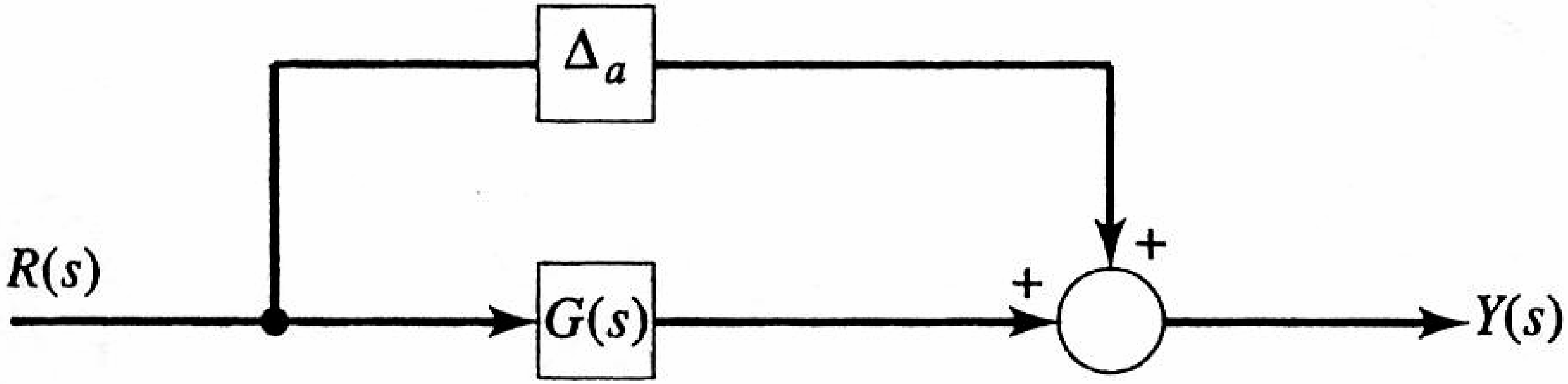
$$\Delta_m(s) = \frac{\tilde{G}(s) - G(s)}{G(s)}$$

Uncertainty

- Example

$$G(s) = \frac{2}{s^2} \quad \tilde{G}(s) = \frac{s^2 + 2s + 2}{s^2(s^2 + s + 1)}$$

$$\Delta_a(s) = \tilde{G}(s) - G(s) = \frac{-1}{s^2 + s + 1}$$

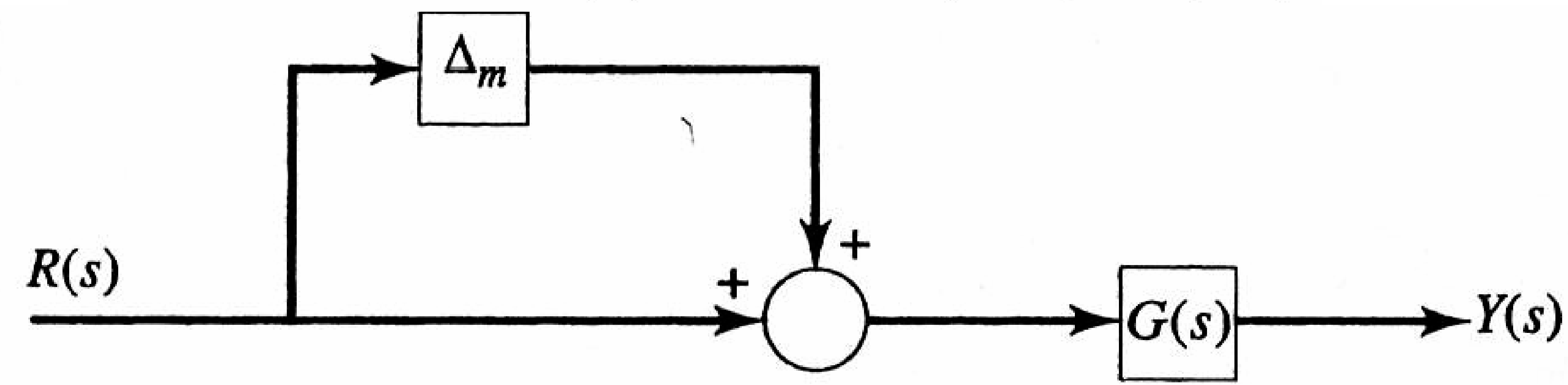


Uncertainty

- Example

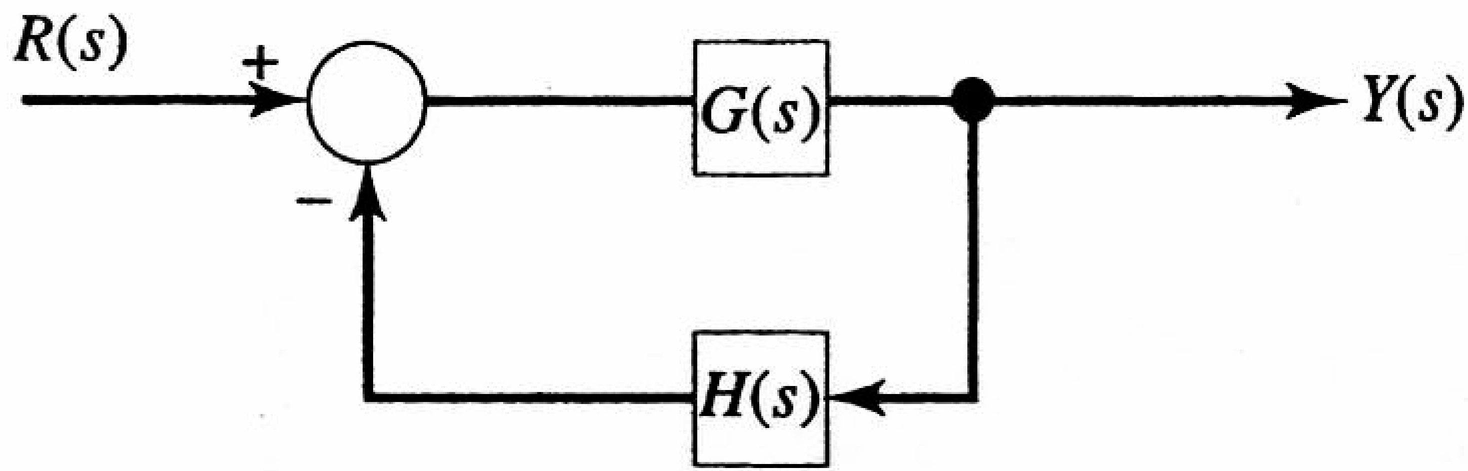
$$G(s) = \frac{2}{s^2} \quad \tilde{G}(s) = \frac{s^2 + 2s + 2}{s^2(s^2 + s + 1)}$$

$$\Delta_m(s) = \frac{\tilde{G}(s) - G(s)}{G(s)} = \frac{-s^2}{2(s^2 + s + 1)}$$



Robust Stability

- Small-Gain Theorem



- Assume G(s),H(s) stable, then the closed-loop system will remain stable if

$$|G(s)H(s)| < 1$$

Robust Stability

- Small-Gain Theorem

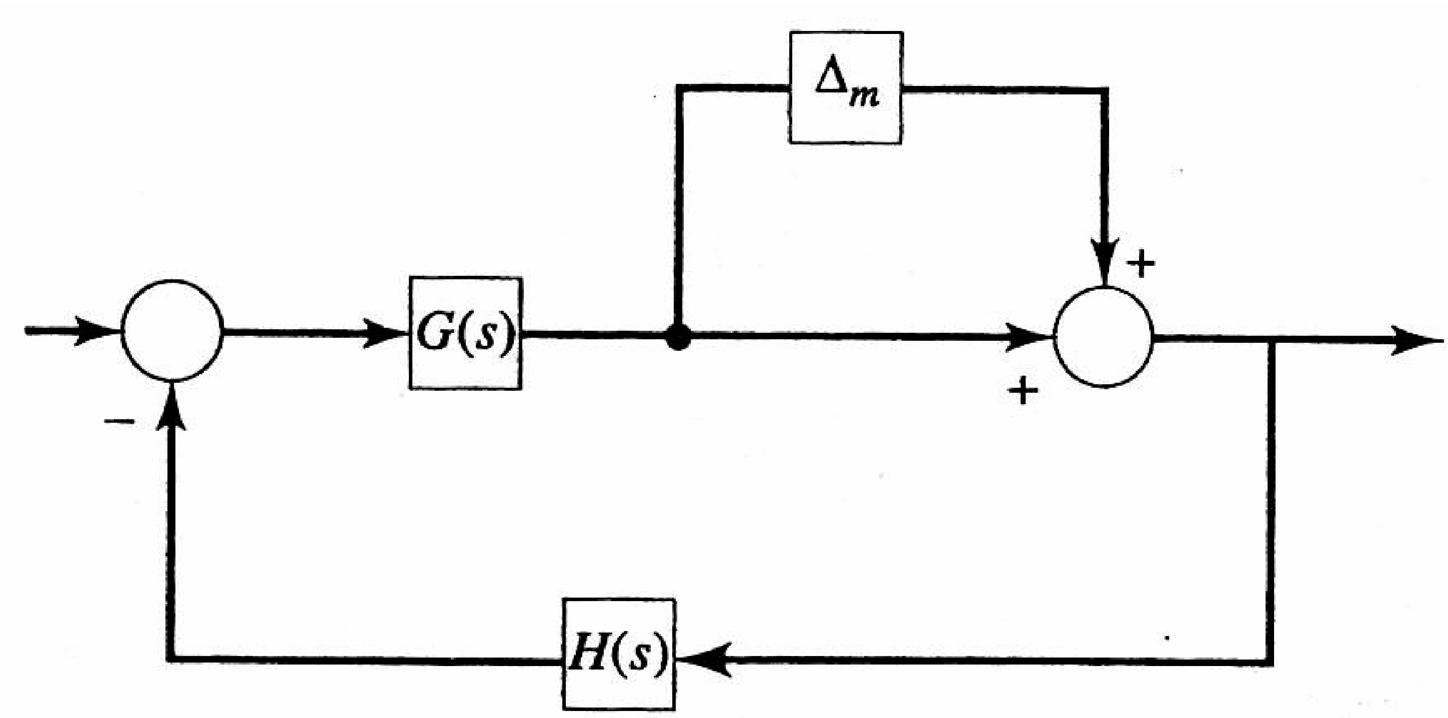
Also, because of the following inequality

$$|G(s)H(s)| \leqslant |G(s)|\,|H(s)|$$

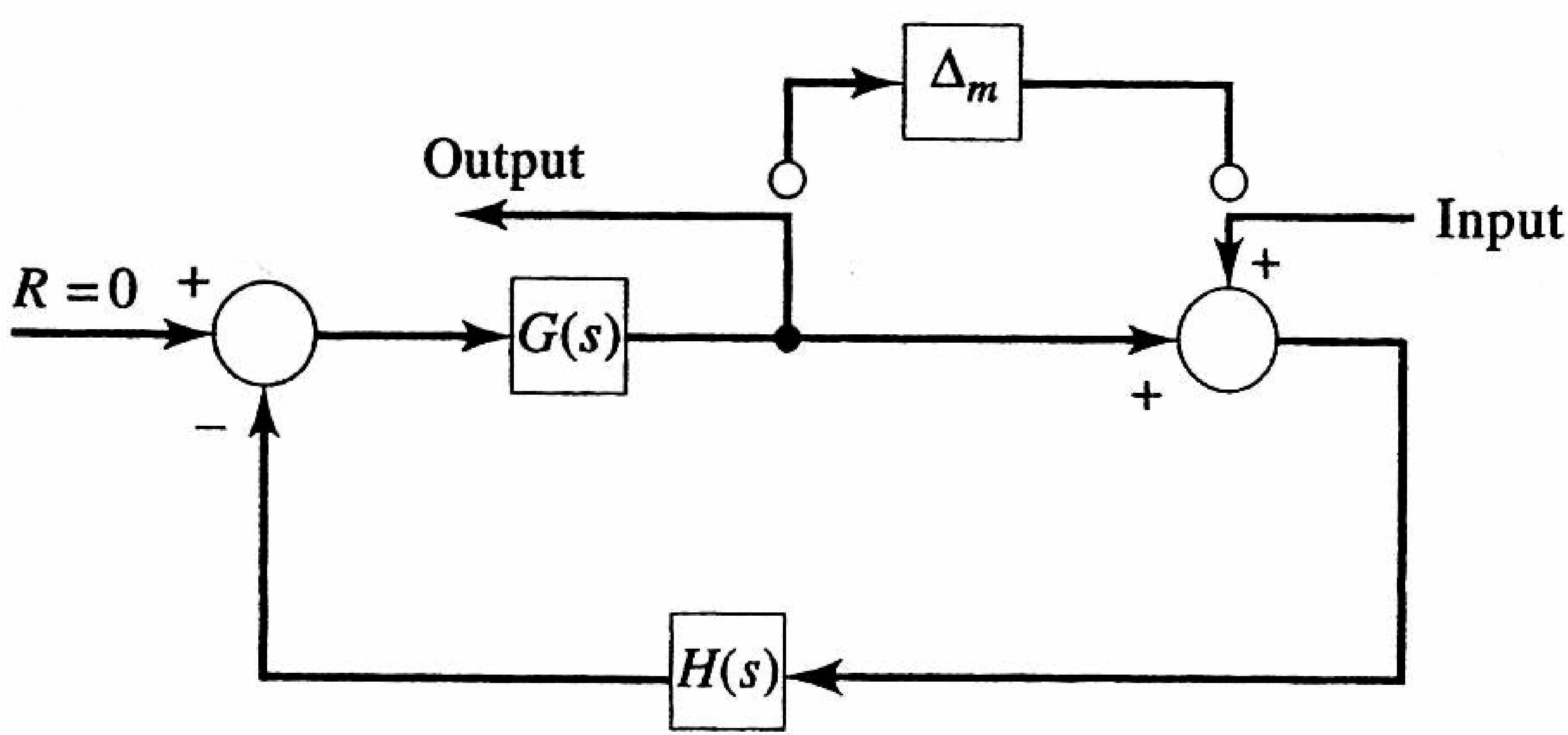
We can also guarantee closed-loop stability if

$$|G(s)|\,|H(s)| < 1$$

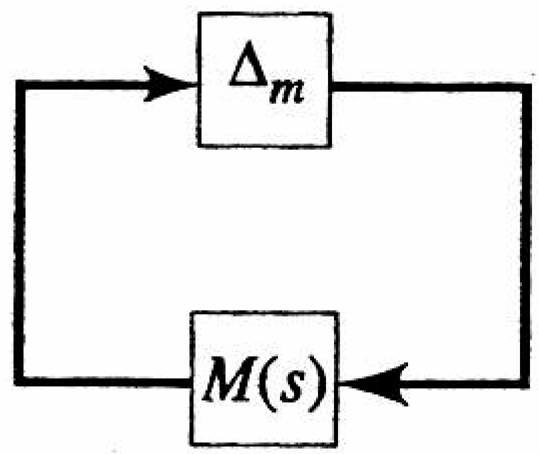
Robust Stability



Robust Stability



Robust Stability



$$M(s) = \frac{-G(s)H(s)}{1 + G(s)H(s)}$$

- The closed-loop system robustly stable if

$$|\Delta_m| < \frac{1}{\left|GH(1 + GH)^{-1}\right|}$$

Robust Stability

- Suppose

$$|\Delta_m| < \gamma$$

- Then the closed-loop system will be stable if

$$|T| < \frac{1}{\gamma} \quad \text{or} \quad |\gamma T| < 1$$

H_∞ Control

- MIMO system : matrix transfer function
- LQG : Time-domain optimization
- Frequency domain optimization
- H_∞ : the space of stable & proper transfer functions

$$\|G\|_\infty = \sup_\omega |G(j\omega)|$$

$$\left\| \frac{1}{s+1} \right\|_\infty = 1$$

- MIMO systems: singular value, SVD

Packed-matrix notation

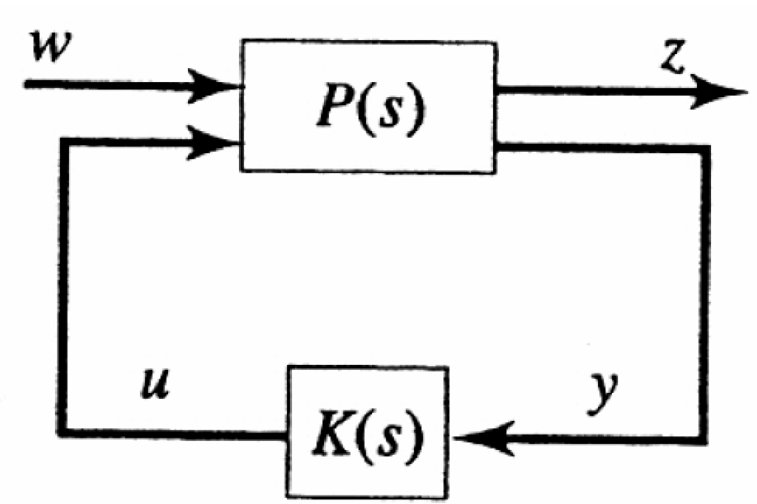
$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

- LQG controller

$$H(s) = \left[\begin{array}{c|c} A - BK - LC & L \\ \hline K & 0 \end{array}\right]$$

Generalized block diagram for H_∞ control



$$z = P_{zw} w + P_{zu} u$$

$$y = P_{yw} w + P_{yu} u$$

$$u = K y$$

Generalized block diagram for H_∞ control

$$y = P_{yw} w + P_{yu} K y$$

$$(I - P_{yu} K) y = P_{yw} w \rightarrow y = (I - P_{yu} K)^{-1} P_{yw} w$$

$$u = K y = K (I - P_{yu} K)^{-1} P_{yw} w$$

$$z = P_{zw} w + P_{zu} K (I - P_{yu} K)^{-1} P_{yw} w = [P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yw}] w$$

$$z = T_{zw} \qquad \text{where } \mathrm{T}_{zw} = P_{zw} + P_{zu} K (I - P_{yu} L)^{-1} P_{yw}$$

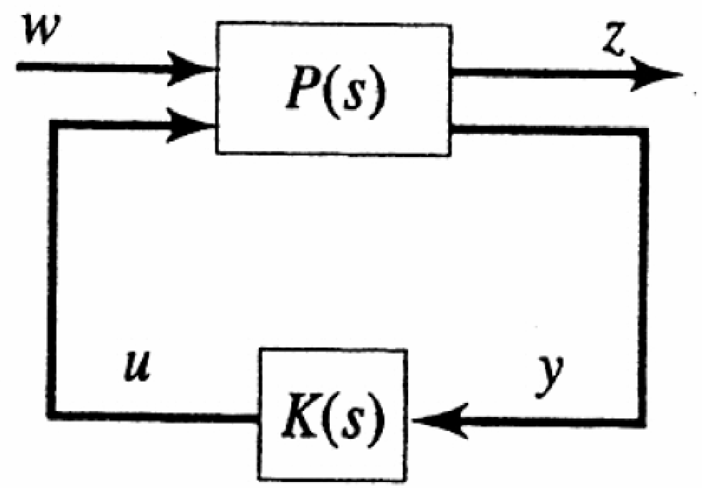
- Closed-loop transfer function

Plant Representation

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{11} w + D_{12} u$$

$$y = C_2 x + D_{21} w + D_{22} u$$



$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

H_∞ Control: Solution

Optimal problem $\min_{K(s)\,\text{stabilizing}} \|T_{zw}\|_\infty$

Standard problem $\text{Find}_{K(s)\,\text{stabilizing}} \|T_{zw}\|_\infty \leqslant \gamma$

- Suboptimal problem

H_∞ Control: Solution

$$u = -K_c\hat{x}$$

and the state estimator is given by

$$\dot{\hat{x}} = A\hat{x} + B_2u + B_1\hat{w} + Z_\infty K_e(y - \hat{y})$$

where

$$\hat{w} = \gamma^{-2}B_1'X_\infty\hat{x}$$

$$\hat{y} = C_2\hat{x} + \gamma^{-2}D_{21}B_1'X_\infty\hat{x}$$

H_∞ Control: Solution

$$K_c = \tilde{D}_{12}(B_2' X_\infty + D_{12}' C_1) \quad \text{where } \tilde{D}_{12} = (D_{12}' D_{12})^{-1}$$

$$K_e = (Y_\infty C_2' + B_1 D_{21}') \tilde{D}_{21} \quad \text{where } \tilde{D}_{21} = (D_{21} D_{21}')^{-1}$$

The term Z_∞ is given by

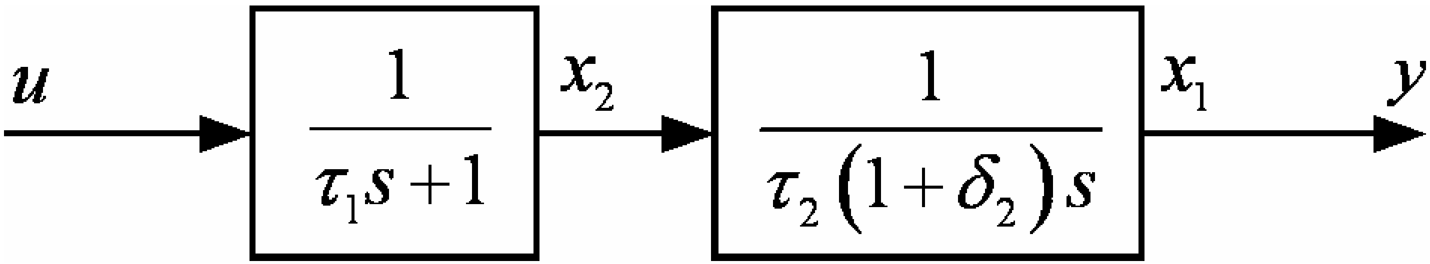
$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

$$X_\infty = \text{Ric} \begin{bmatrix} A - B_2 \tilde{D}_{12} D_{12}' C_1 & \gamma^{-2} B_1 B_1' - B_2 \tilde{D}_{12} B_2' \\ -\tilde{C}_1' \tilde{C}_1 & -(A - B_2 \tilde{D}_{12} D_{12}' C_1') \end{bmatrix}$$

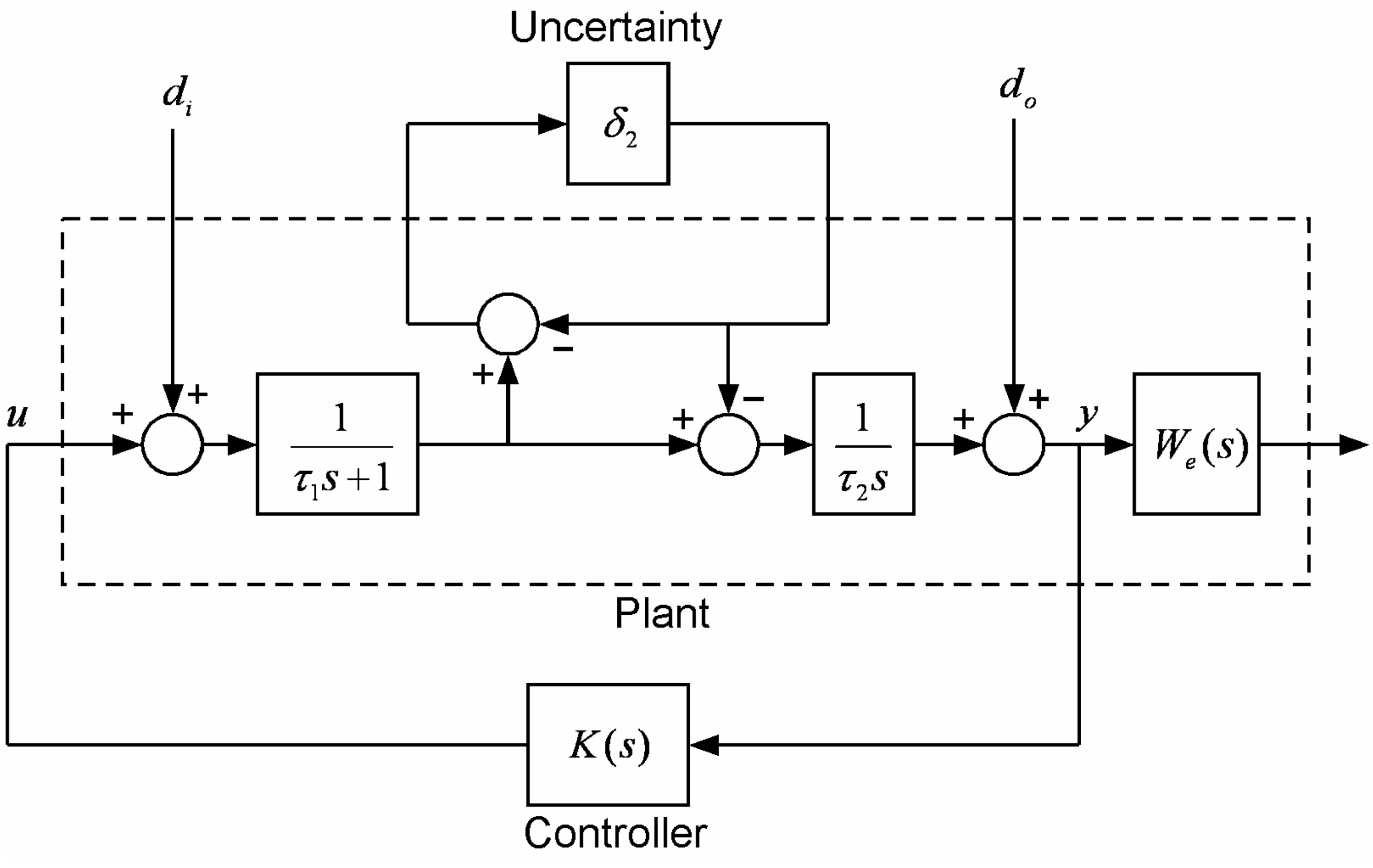
$$Y_\infty = \text{Ric} \begin{bmatrix} (A - B_1 D_{21}' \tilde{D}_{21} C_2)' & \gamma^{-2} C_1' C_1 - C_2' \tilde{D}_{21} C_2 \\ -\tilde{B}_1 \tilde{B}_1' & -(A - B_1 D_{21}' \tilde{D}_{21} C_2) \end{bmatrix}$$

where $\tilde{C}_1 = (I - D_{12} \tilde{D}_{12} D_{12}') C_1$ and $\tilde{B}_1 = B_1 (I - D_{21}' \tilde{D}_{21} D_{21})$.

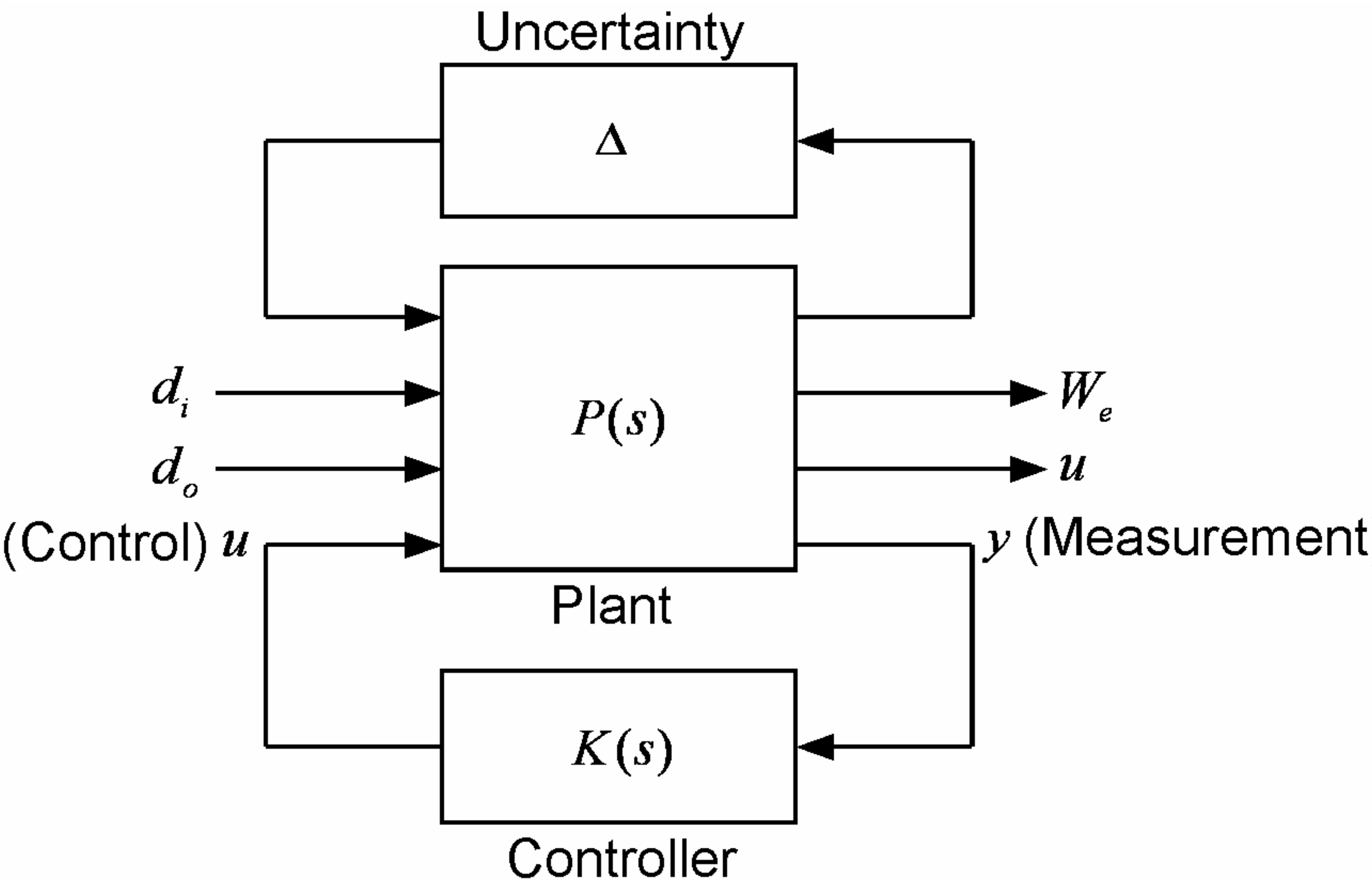
Dynamic Simulator



Dynamic Simulator



Dynamic Simulator



LQG controller

$$u = -K\hat{x}$$

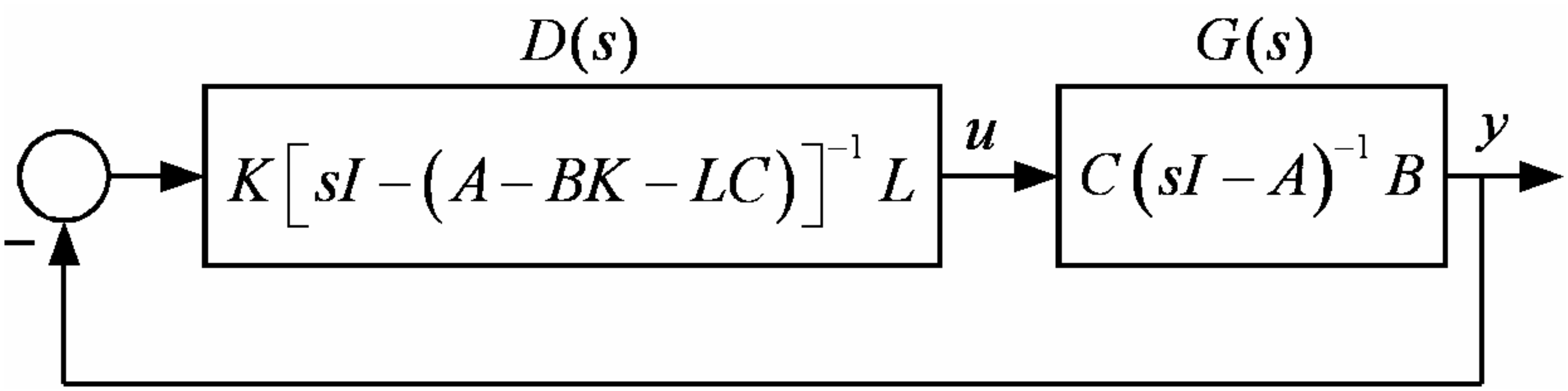
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

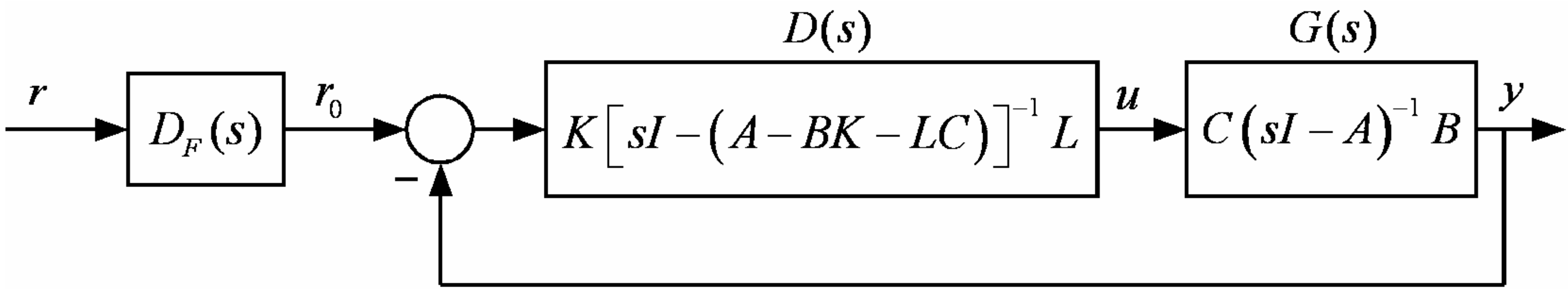
$$= (A - BK)\hat{x} + L(y - C\hat{x})$$

$$= (A - BK - LC)\hat{x} + Ly$$

$$D(s) = -K(sI - A + BK + LC)^{-1}L$$

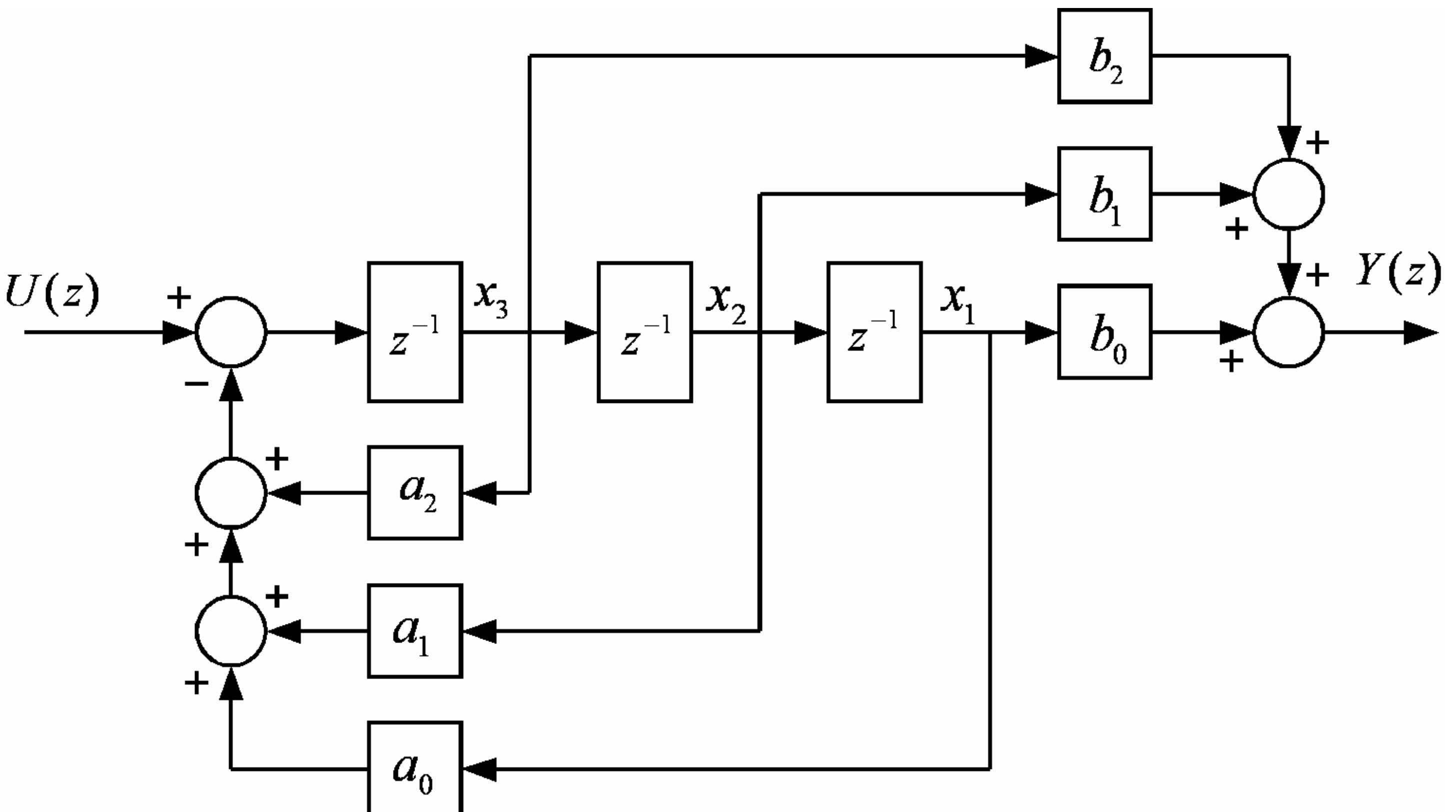
LQG controller





Controller Canonical Form

$$D(z) = \frac{Y(z)}{U(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$



Controller Canonical Form

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} x(k)$$

Controller Canonical Form

$$D(z) = \frac{Y(z)}{U(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0} + d_0$$

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} x(k) + d_0 u(k)$$

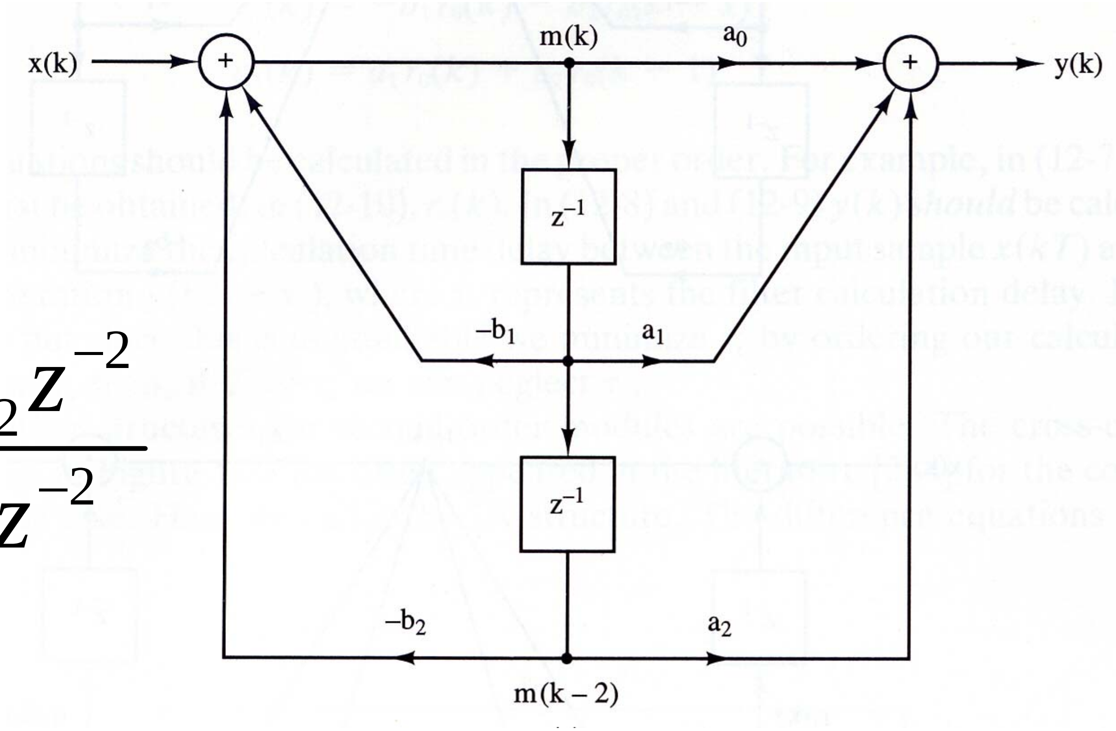
Canonical Form Implementation

```
x1=x2_old;
x2=x3_old;
x3=-0.9108*x1_old+0.9464*x2_old+0.9629*x3_old+(ref-x);
y=0.0945*x1_old-0.1272*x2_old+0.0302*x3_old+2.383*(ref-x);

x1_old=x1;
x2_old=x2;
x3_old=x3;
```

Second-order Modules

$$D(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$



$$m(k) = x(k) - b_1 m(k-1) - b_2 m(k-2)$$

$$y(k) = a_0 m(k) + a_1 m(k-1) + a_2 m(k-2)$$

Cascade Form Implementation

```
m1=2.383*(ref-x)-0.9727*m1_old;
y1=m1+m1_old;

m2=y1+0.9844*m2_old;
y2=m2-0.995*m2_old;

m3=y2+0.9512*m3_old;
y3=m3-0.9553*m3_old;

y=y3;
m1_old=m1;
m2_old=m2;
m3_old=m3;
```